Ex:

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T Q x + c^T x, \quad Q > 0 \qquad \text{(Primal)}$$

$$\text{s.t. } Ax \leq b \qquad A \in \mathbb{R}^{r \times n}$$

$$r << n$$

100 k

n is very large

↳ Computational / Bandwidth requirement on communication channel

Can we come up with a simpler optimization problem?

↳ Lesser bandwidth requirement (scales with r and not n)

↳ Can the constraints be simplified further

↳ Yes

Primal ⟹ Dual optimization Problem

↳ Weak duality

⇓ Conditions under which

(Strong duality)

Lagrangian dual function

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* Standard (primal) convex optimization problem:

$$p^* := \begin{cases} \min_{x \in \mathbb{R}^n} f(x) \\ \text{s.t. } h_i(x) \leq 0 \quad \forall\, i \in \{1, 2, \ldots, m\} \\ \quad\;\; l_j(x) = 0 \quad \forall\, j \in \{1, 2, \ldots, r\} \end{cases}$$

(*) **Assumption:** $f, \{h_i\}, \{l_j\}$ are convex and p^* is finite.

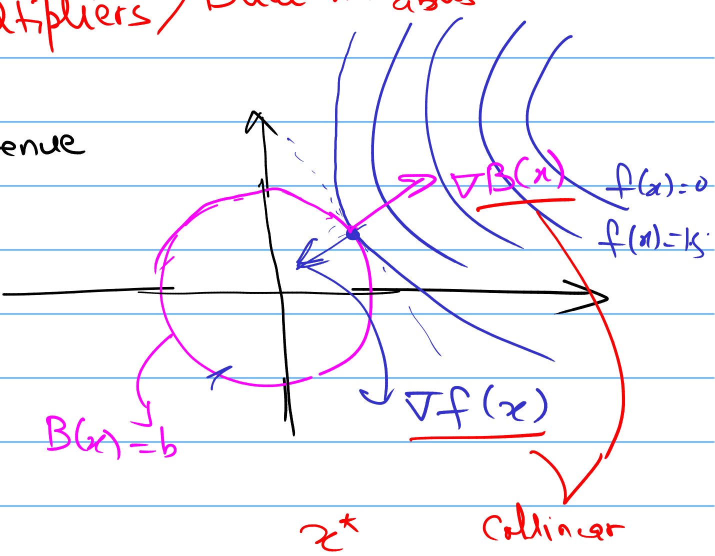
We define Lagrangian as:

$$\underline{L}(x,\lambda,\nu) := f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{r} \nu_j l_j(x)$$

$$\left.\begin{array}{l}\lambda \in \mathbb{R}^m \\ \nu \in \mathbb{R}^r\end{array}\right\}$$ Lagrange Multipliers / Dual variables.

Ex: $\min_{x \in \mathbb{R}^n} f(x)$ ← Loss in revenue

s.t. $B(x) = b$ ← budget



$$\nabla f(x) = -\nu \nabla B(x)$$

$$\boxed{\nabla f(x) + \nu \nabla B(x) = 0}$$

$$\boxed{\nabla f(x^*) + \nu^* \nabla B(x^*) = 0}$$

$$\boxed{B(x^*) = b}$$

$*$ $$L(x,\nu) = f(x) + \nu(B(x) - b)$$

✓ $$\boxed{\nabla f(x^*) + \nu^*(\nabla B(x^*)) = 0}$$ [Gradient w.r.t x]

✓ $$\boxed{B(x^*) = b}$$

$*$ Lagrangian dual function:

$$g(\lambda,\nu) := \min_{x \in \mathbb{R}^n} f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{r} \nu_j l_j(x)$$

$*$ Lower bound property:

If $\lambda_i \geq 0$, then $\boxed{p^* \geq g(\lambda,\nu)}$

Proof: Let us fix $\bar{x} \in \mathcal{X}$

$$\underline{f(\bar{x})} \ge f(\bar{x}) + \underbrace{\sum_{i=1}^{m} \lambda_i \underbrace{h_i(\bar{x})}_{\le 0}}_{\le 0} + \sum_{j=1}^{\gamma} \nu_j \underbrace{l_j(\bar{x})}_{=0}$$

$$= L(\bar{x}, \lambda, \nu)$$

$$\ge \underbrace{\min_{x \in \mathbb{R}^n} L(x, \lambda, \nu)}_{g(\lambda, \nu)}$$

$$f(\bar{x}) \ge g(\lambda, \nu) \quad \forall \bar{x} \in \mathcal{X}$$

$$\underbrace{\min_{\bar{x} \in \mathcal{X}} f(\bar{x})} \ge g(\lambda, \nu)$$

$$\boxed{p^* \ge g(\lambda, \nu)}$$ ——— ☒

Remark: This holds true even if $f, \{h_i\}$ and $\{l_j\}$ are non-convex.

——x——x——x——x——x——

$$p^* \ge \underbrace{\max_{\substack{\lambda \ge 0 \\ \lambda, \nu}} g(\lambda, \nu)}_{d^*}$$

$p^* \ge d^*$

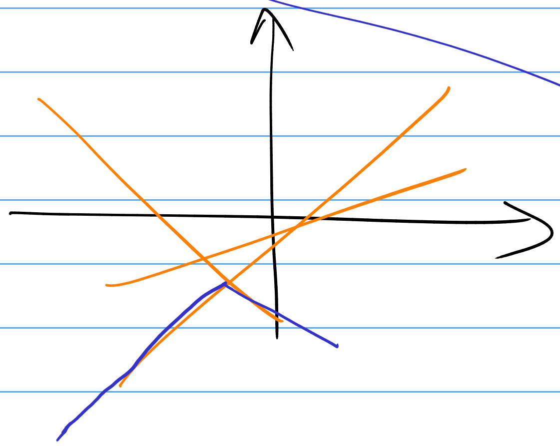
[Weak duality]

$p^* = d^*$ → Strong duality

$$g(\lambda, \nu) = \underbrace{\min_{x \in \mathbb{R}^n} \left\{ f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{\gamma} \nu_j l_j(x) \right\}}_{(\lambda, \nu)}$$

Pointwise - minimization of affine functions in (λ, ν)

→ is always concave even if $f, \{h_i\}$ and $\{l_j\}$ are non-convex



* Duality Gap:

$$\underline{G} := p^* - d^* \geq 0$$

* Lagrangian dual problem:

$$d^* := \max_{\substack{(\lambda,\nu)\\ \lambda\geq 0}} g(\lambda,\nu)$$

* Example of duality gap:

$$\min_{x\in\mathbb{R}^2} \boxed{e^{x_2}} = 1$$

$$\text{st} \quad \|x\|_2 \leq x_1$$

$$\mathcal{X} = \left\{ (x_1,x_2) : \sqrt{x_1^2+x_2^2} \leq x_1 \right\}$$

$$= \left\{ (x_1,x_2) : x_1 \geq 0 , \underline{x_2 = 0} \right\}$$

$$\boxed{p^* = 1} \quad \text{since } e^{x_2} = e^0 = 1$$

$$g(\lambda) = \min_{x\in\mathbb{R}^2} \left\{ \underbrace{f(x)}_{\substack{e^{x_2}\\ \geq 0}} + \lambda \underbrace{\left(\sqrt{x_1^2+x_2^2} - x_1\right)}_{\geq 0} \right\}$$

$$\boxed{d^* \geq 0} \qquad d^* := \max_{\lambda\geq 0} g(\lambda) \geq 0$$

$$\underline{\eta(x_1,x_2)} = \sqrt{x_1^2+x_2^2} - x_1 = \frac{x_2^2}{\sqrt{x_1^2+x_2^2}+x_1}$$

$$\text{If } x_1 = x_2^4 \qquad \eta(x_1,x_2) \leq \frac{1}{x_2^2} \xrightarrow[x_2\to-\infty]{} 0$$

$$g(\lambda) = \underline{\min_{x \in \mathbb{R}^2}} \left(e^{x_2} + \lambda \eta(x_1, x_2) \right)$$

$$\le \min_{x_1 = x_2^4} \left(\underline{e^{x_2}} + \lambda \underbrace{\eta(x_1, x_2)}_{\downarrow 0} \right)$$

A $\underline{x_2 \to -\infty}$ $\quad \to 0$

$\underline{g(\lambda) \le 0}$ $\quad \boxed{d^* \le 0}$

$$\boxed{d^* = 0}$$

$$\boxed{G = p^* - d^* = 1 > 0} \longrightarrow \exists \text{ duality gap}$$

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* Strong duality: $G = 0$ or $p^* = d^*$

Slater's condition: If $\exists\, \bar{x}$ which is strictly feasible,

$h_i(\bar{x}) < 0 \quad \forall\, i \in \{1, 2, \dots, m\}$

then Assumption $(*)$ + Strict feasibility

$$\Downarrow$$

Strong duality.

Remark: For linear inequality constraints, strong duality holds even without strict feasability.

↳ Can work with dual optimization problems.

↳ If strong duality holds, then KKT conditions, which are always sufficient, also become necessary.

* How should we interpret dual variables?

$$\left.\begin{array}{l}\min\limits_{x\in\mathbb{R}^n} f(x) \\ \text{s.t.} \quad B(x)=b\end{array}\right\} \longrightarrow \nu^*$$

$$L(x,\nu) = f(x) + \nu(B(x)-b)$$

$$B(x^*)=b$$

$$\nabla f(x^*) + \nu^* \nabla B(x^*) = 0$$

Let us say M^* is the optimal value.

$$M^* = f(x^*) = L(x^*,\nu^*)$$

$$x^*(b) \qquad \nu^*(b)$$

$$\frac{dM^*}{db} = \frac{d\,L(x^*,\nu^*)}{db} = \frac{\partial L(x^*,\nu^*)}{\partial b} + \cancelto{0}{\frac{\partial L}{\partial x^*}}\frac{\partial x^*}{\partial b} + \cancelto{0}{\frac{\partial L}{\partial \nu^*}}\frac{\partial \nu^*}{\partial b}$$

$$= -\nu^*$$

↳ Incremental cost variable.

Ex:

$$\left.\begin{array}{l}\min\limits_{x\in\mathbb{R}^n} \frac{1}{2}x^TQx + c^Tx \\ \text{s.t.} \quad Ax \le b\end{array}\right\} \longrightarrow \text{Primal form}, \quad Q \succ 0$$

Strong duality holds $p^* = d^*$

$$g(\lambda) = \min_{x\in\mathbb{R}^n} \left\{ \frac{1}{2}x^TQx + c^Tx + \lambda^T(Ax-b) \right\}$$

$$Qx^* + c + A^T\lambda = 0$$

$$x^* = -Q^{-1}(c + A^T\lambda)$$

$$\boxed{\max_{\lambda \geq 0} g(\lambda) = -\frac{1}{2}\lambda^T P \lambda - a^T \lambda}$$

$$P := AQ^{-1}A^T$$

$$a := b + AQ^{-1}c$$