* f is closed & convex

f is μ-sc $\iff \nabla f^*$ is $\frac{1}{\mu}$-Lipshitz.

(f^* is $\frac{1}{\mu}$-smooth)

($\Rightarrow$): f is μ-sc.

$$\|\nabla f(x) - \nabla f(y)\| \geq \mu \|x - y\|$$

$$x \in \nabla f^*(y) \iff y \in \nabla f(x)$$

$$s_x := \nabla f(x) \Rightarrow x = \nabla f^*(s_x)$$

$$\|s_x - s_y\| \geq \mu \|\nabla f^*(s_x) - \nabla f^*(s_y)\|$$

$$\|\nabla f^*(s_x) - \nabla f^*(s_y)\| \leq \frac{1}{\mu} \|s_x - s_y\|$$

When do we guarantee avg. consensus?

- $\hookrightarrow$ A is row-stochastic $\Rightarrow \lambda = 1$ is an eigenvalue of A.
- $\hookrightarrow$ Underlying graph is connected $\Rightarrow \lambda = 1$ is a simple "
- $\hookrightarrow$ A is symmetric $\Rightarrow$ Avg. consensus

* Sinkhorn-Knopp's Algorithm

$$A = \begin{bmatrix} * & 0 & * & * & 0 & \cdots & * \\ \cdot & & \vert & & & & \\ \vert & & & & & & \\ \cdot & & & & & & \end{bmatrix} \begin{array}{l} \to R_1 \\ \to R_2 \end{array} \in \mathbb{R}^{N \times N}$$

Step 1: Row-normalization (Normalization by row-sum)

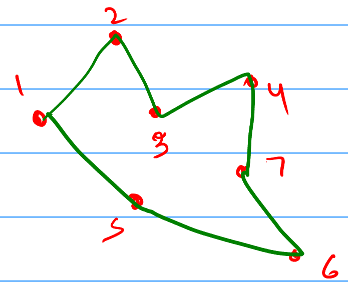
$$A^+ := \begin{bmatrix} \frac{*}{R_1} & 0 & \frac{*}{R_1} & \cdots \\ \frac{\circledast}{R_2} & - & - & \cdot \\ & & & \end{bmatrix}$$

Step 2: Column-normalization (Normalization by column-sum)

$$A^{+} \quad \begin{bmatrix} \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \cdots \\ C_1 & C_2 & C_3 & \end{bmatrix}$$

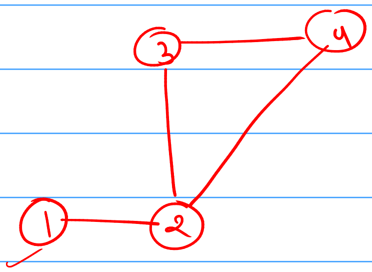
$$A^{++} = \frac{A^{+}}{[C_1 \; C_2 \; \cdots \; C_N]}$$

Repeat Steps 1 and 2 until convergence $\Rightarrow$ Doubly stochastic matrix.



$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & & & & & \\ 0 & & & & & \\ 0 & & & & & \\ 1 & & & & & \\ 0 & & & & & \\ 0 & & & & & \end{bmatrix}$$

* Example:



$$\underbrace{A = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_4(k+1) \end{bmatrix}}$$

$$x_1(k+1) \leftarrow \frac{x_1(k) + x_2(k)}{2}$$

$$x_2(k+1) \leftarrow \frac{x_1(k) + x_2(k) + x_3(k) + x_4(k)}{4}$$

$$x_3(k+1) \leftarrow \frac{x_2(k) + x_3(k) + x_4(k)}{3}$$

$$x_4(k+1) \leftarrow \frac{x_2(k) + x_3(k) + x_4(k)}{3}$$

* Metropolis weighting scheme:

Markov Chain

$$a_{ij} = \begin{cases} \dfrac{1}{1+\max\{d_i, d_j\}} & , \text{if } i \neq j \\ 1 - \sum\limits_{j \neq i} a_{ij} & , \text{if } i = j \end{cases}$$

$$a_{12} = \frac{1}{1+3} = \frac{1}{4} \; ; \quad a_{11} = 3/4$$

$$a_{21} = \frac{1}{4}, \; a_{23} = \frac{1}{4}, \; a_{24} = \frac{1}{4} \; ; \quad a_{22} = 1/4$$

$$a_{32} = \frac{1}{4}, \; a_{34} = \frac{1}{3} \; ; \quad a_{33} = \frac{5}{12}$$

$$a_{42} = \frac{1}{4}, \; a_{43} = 1/3 \; ; \quad a_{44} = \frac{5}{12}$$

$$A = \begin{bmatrix} 3/4 & 1/4 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/4 & 5/12 & 1/3 \\ 0 & 1/4 & 1/3 & 5/12 \end{bmatrix}$$

$x(k+1) = A\,x(k) \longrightarrow$ guarantees avg. consensus with Metropolis weighting.

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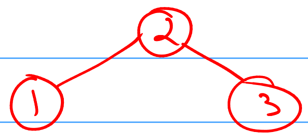
* Standard consensus algorithm in continuous-time.

* FxTS variant of consensus algorithm in continuous-time.



$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \dot{x} = \cancel{A}x$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L = D - A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$L \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\bar{x}} = \begin{bmatrix} x_1 - x_2 \\ 2x_2 - x_1 - x_3 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} (x_1 - x_2) \\ (x_2 - x_1) + (x_2 - x_3) \\ (x_3 - x_2) \end{bmatrix}$$

$L\bar{x} \longrightarrow$ Local computations.

$$\boxed{\dot{x} = -Lx} \longrightarrow \text{standard Consensus algorithm.}$$

Equilibrium of this dynamical system.

$$\Rightarrow x^* = \alpha \mathbb{1}_N$$

$$\sum_{i=1}^{N} \dot{x}_i = -\sum_{i=1}^{N} \sum_{j=1}^{N} L_{ij} x_j = 0$$

$$\left[\because \sum_{i=1}^{N} L_{ij} = 0 \quad \forall j \right]$$

$$\mathbb{1}^T L = \bar{0}$$

$$\sum_{i=1}^{N} \dot{x}_i = 0$$

$$\Rightarrow \sum_{i=1}^{N} x_i(t) = \text{constant} = \sum_{i=1}^{N} x_i(0) = N x^*$$

$$x^* = \frac{1}{N} \sum_{i=1}^{N} x_i(0)$$

$$\Downarrow$$

Average Consensus

$$x(t) = e^{-Lt} x(0) \;;$$ L is a PSD matrix with 0 being a simple eigenvalue if graph is connected.

$$\boxed{\dot{x}_i = -\sum_{j \in N_i} (x_i - x_j)}$$

—— x —— x —— x —— x ——

* FxTS Consensus scheme:

Results: (i) $\sum_{i=1}^{N} \sum_{j\in N_i} \mathrm{sign}(x_i - x_j) = 0$

(ii) $\sum_{i=1}^{N} \sum_{j\in N_i} e_i^T w(x_{ij}) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j\in N_i} e_{ij}^T w(x_{ij})$

where, $w(\cdot)$ is an odd function

(iii) $z_i > 0$

$$\sum_{i=1}^{N} z_i^{p} \geq \left(\sum_{i=1}^{N} z_i\right)^{p} \quad \text{if } p\in(0,1)$$

$$\sum_{i=1}^{N} z_i^{p} \geq N^{1-p} \left(\sum_{i=1}^{N} z_i\right)^{p} \quad \text{if } p>1$$

(iv) if $\mathbb{1}^T x = 0$, $x^T L x \geq \lambda_2(L) \|x\|^2$

* Defn: $\mathrm{sgn}^{\mu}(x) := x\,\|x\|^{\mu-1}$ $\mu \geq 0$

when $\mu=0$, $\mathrm{sgn}^0(x) = \frac{x}{\|x\|} = \mathrm{sign}(x)$

FxTS consensus scheme:

$$\boxed{\begin{gathered}\dot{x}_i = -\sum_{j\in N_i} \mathrm{sgn}^{\mu_1}(x_i - x_j) - \sum_{j\in N_i} \mathrm{sgn}^{\mu_2}(x_i - x_j) \\ \text{with } \left.\begin{array}{l}\mu_1 \in (0,1) \\ \mu_2 > 1\end{array}\right] \text{— odd} \\ 1/3 \qquad 5/3\end{gathered}}$$

Proof: $\sum_{i=1}^{N} \dot{x}_i = 0 \implies \sum_{i=1}^{N} x_i(t) = \text{constant} = \sum_{i=1}^{N} x_i(0)$

$$x_c = \frac{1}{N} \sum_{i=1}^{N} x_i(0)$$

$\tilde{x}_i = x_i - x_c \longrightarrow$ Want this to converge to 0 for all i

Choice of Lyapunov candidate:

$$V = \frac{1}{2}\sum_{i=1}^{N} \tilde{x}_i^T \tilde{x}_i$$

$$\dot{\tilde{x}}_i = \dot{x}_i - \cancel{\dot{x}_c}^{\,0} = \dot{x}_i$$

Independent of topology

This scheme works for time varying graphs

Taking time-derivative of V:

$$\dot{V} = \sum_{i=1}^{N} \tilde{x}_i^T \dot{\tilde{x}}_i$$

$$= \sum_{i=1}^{N} \tilde{x}_i^T \dot{x}_i$$

$$= -\sum_{i=1}^{N} \tilde{x}_i^T \left[\sum_{j\in N_i} \operatorname{sgn}^{\mu_1}(\tilde{x}_i - \tilde{x}_j) + \sum_{j\in N_i} \operatorname{sgn}^{\mu_2}(\tilde{x}_i - \tilde{x}_j) \right]$$

$$= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j\in N_i} \tilde{x}_{ij} \left[\operatorname{sgn}^{\mu_1}(\tilde{x}_{ij}) + \operatorname{sgn}^{\mu_2}(\tilde{x}_{ij}) \right]$$

$$= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j\in N_i} \left[\|\tilde{x}_{ij}\|^{1+\mu_1} + \|\tilde{x}_{ij}\|^{1+\mu_2} \right]$$

$$= -\frac{1}{2}\sum_{i=1}^{N} \left(\sum_{j=1}^{N} \Big(\underbrace{a_{ij}\|\tilde{x}_{ij}\|^2}_{\eta_{ij}}\Big)^{\frac{1+\mu_1}{2}} + \sum_{j=1}^{N} \Big(\underbrace{a_{ij}\|\tilde{x}_{ij}\|^2}_{\eta_{ij}}\Big)^{\frac{1+\mu_2}{2}} \right]$$

$$= -\frac{1}{2}\left(\sum_{i,j=1}^{N} \left(a_{ij}\|\tilde{x}_{ij}\|^2\right)^{\frac{1+\mu_1}{2}} + \sum_{i,j=1}^{N} \left(a_{ij}\|\tilde{x}_{ij}\|^2\right)^{\frac{1+\mu_2}{2}} \right]$$

$$\leq -\frac{1}{2}\left[\Big(\underbrace{\sum_{i,j=1}^{N} \eta_{ij}}\Big)^{\frac{1+\mu_1}{2}} + (N^2)^{\frac{1-\mu_2}{2}} \Big(\sum_{i,j=1}^{N} \eta_{ij}\Big)^{\frac{1+\mu_2}{2}} \right]$$

$$\sum_{i,j=1}^{N} a_{ij}\|\tilde{x}_{ij}\|^2 \qquad \tilde{x}^T L x = \sum_{i,j=1}^{N} a_{ij}\|\tilde{x}_{ij}\|^2$$

$$\hookrightarrow \tilde{x}^T L x \geq \lambda_2 \|\tilde{x}_{ij}\|^2$$

$$\mathbf{1}^T \tilde{x} = 0$$

$$\leq -\frac{\lambda_2}{2}\left[\Big(\underbrace{4\|\tilde{x}\|^2}_{2V}\Big)^{\frac{1+\mu_1}{2}} + (N^2)^{\frac{1-\mu_2}{2}} \Big(\underbrace{\|\tilde{x}\|^2}_{2V}\Big)^{\frac{1+\mu_2}{2}} \right]$$

$\tilde{x}^T L x$

$L \otimes I_N$

$$\Rightarrow \quad \dot V \le -\underbrace{\frac{\Lambda_2(L)^{\frac{1+\mu_1}{2}}\, 2^{\frac{1+\mu_1}{2}}}{2}}_{c_1} V^{\frac{1+\mu_1}{2}} - \underbrace{\frac{\Lambda_2(L)^{\frac{1+\mu_2}{2}}\, 2^{\frac{1+\mu_2}{2}}\, N^{1-\mu_2}}{2}}_{c_2} V^{\frac{1+\mu_2}{2}}$$

$$\dot V \le -c_1 V^{\alpha_1} - c_2 V^{\alpha_2} \qquad \alpha_1 \in (0,1)$$

$$\alpha_2 > 1$$

$$T = \frac{1}{c_1(1-\alpha_1)} + \frac{1}{c_2(\alpha_2-1)}$$

$\Rightarrow$ Scheme converges in a fixed-time.

$\therefore$ This is a fixed-time convergent average consensus scheme.